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STRATEGIC DIVISION NOTE

SDN 77-1

A MODEL FOR EXAMINING A SEARCH AND ATTACK OPERATION AGAINST A GROUP OF MOVING QUARRIES

May 1977

Prepared by
Martin Stern

Approved by
J. W. Seelig, Division Manager

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5613 LEESBURG PIKE, FALLS CHURCH, VIRGINIA 22041

TO: DDC

DATE: 23 June 1977

FROM: J. W. Seelig, Manager, Strategic Division

J.W.S.

SUBJECT: ERRATUM TO ANSER STRATEGIC DIVISION NOTE SDN 77-1

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Please note the following correction to ANSER Strategic Division Note SDN 77-1, A Model for Examining a Search and Attack Operation Against a Group of Moving Quarries, which you received recently. In Figure 2, page 20, the curves representing 250 weapons and 1,000 weapons should be interchanged.

STRATEGIC DIVISION NOTE

SDN 77-1

A MODEL FOR EXAMINING A SEARCH AND ATTACK OPERATION AGAINST A GROUP OF MOVING QUARRIES

May 1977

Prepared by
Martin Stern

Approved by
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anser analytic services inc.
5613 LEESBURG PIKE FALLS CHURCH, VIRGINIA 22041

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FIGURES

I. INTRODUCTION

This note presents a mathematical model of an operation in which a pursuer attempts to locate and destroy a group of quarries. The model was developed to aid in the study of the credibility of current and forecasted threats to the survival of a ballistic missile submarine force. The credibility of such threats was previously examined in another ANSER work, "A Model for Examining the Feasibility of a Nuclear Barrage Attack on a Submarine Force."[1] This note, however, makes a more general set of assumptions about the characteristics of the pursuer, the quarries, and the weaponry used to attack the quarries.

Nothing inherent in the model limits its use to the study of a submarine scenario. In fact, throughout this note the nonspecific terms "pursuer" and "quarry" are used to emphasize the generality of the model. Any scenario that fits into the following framework can be modeled (although one ought to consider the validity of the model's assumptions and approximations for the particular scenario being studied). A pursuer with imperfect searching and tracking abilities searches for a predetermined length of time for a group of quarries, each quarry moving in a region distinct from that of any other quarry. Different searching and tracking characteristics may be specified for each quarry; however, these characteristics remain fixed for each quarry throughout the search. At the end of the search, the pursuer allocates his weapons to maximiz the expected number of quarries to be destroyed and launches the attack. Using a combination of analytic, numerical, and Monte Carlo methods, the model (currently implemented through a FORTRAN program) calculates the expected number and the variance of the number of quarries destroyed.

The first part of this note defines the model and explains the basic assumptions. It then develops the mathematical relationships between the effectiveness of the attack on the quarries and the values of the parameters that drive the model. Initially, it considers only a single-quarry operation. The results obtained for the single-quarry case are then applied to solve the multiple-quarry case. The second part of this note demonstrates how the model can be used to study the vulnerability of a group of quarries as a function of the parameters of the model. Specifically, it presents an example in which a submarine force is threatened by a nuclear barrage attack. By varying the model's parameters, it examines the sensitivity to these parameters of the expected number of submarines destroyed. The example is not meant to be exhaustive or necessarily realistic, only instructive.

II. MODEL

A. Single-Quarry Operation

Assume a single quarry moves with maximum speed s on an unbounded plane. Suppose a pursuer will carry out an operation to locate and destroy this quarry. The operation will consist of a search, which will last exactly T hours, followed by an attack. During the search, he may detect and lose the quarry one or more times, or he may not detect the quarry at all. When the pursuer has no contact with the quarry, the probability of making contact during any time interval within the search period will be a function solely of the length t of the interval. Denote this probability by $R_1(t)$. When the pursuer has contact with the quarry, the probability of losing contact during any time interval within the search period will also be a function solely of the length t of that interval. Denote this probability by $R_2(t)$.

Whenever the pursuer has contact with the quarry, the error distribution of his knowledge of the quarry's true position will be bivariate gaussian with covariance matrix vI, where v is a scalar and I is the identity matrix. Therefore, the probability density function of the random vector U that gives the vector difference $(\Delta X, \Delta Y)$ between the true location of the quarry and the location determined by the pursuer is

$$h(\Delta X, \Delta Y) = \frac{1}{2\pi v} \exp\left(- \frac{\Delta X^2 + \Delta Y^2}{2v}\right) .$$

At the conclusion of the search, the pursuer may or may not have contact with the quarry. In either case, if t denotes the number of hours between the last contact and the end of the search (assuming contact was made at least once during

the search), then the probability distribution of the quarry's final location from his location t hours before will be uniform within a disc of radius st. That is, the probability density function of the random vector V that gives the vector difference $(\Delta X, \Delta Y)$ between the quarry's final true position and his true position when the pursuer last lost contact is

$$g_t(\Delta X, \Delta Y) = \begin{cases} \dfrac{1}{\pi (st)^2} & \text{for } (\Delta X^2 + \Delta Y^2)^{1/2} \leq st \\ 0 & \text{for } (\Delta X^2 + \Delta Y^2)^{1/2} > st \end{cases} .$$

First we will derive an expression for $F(t)$, the cumulative distribution function of the random variable Z whose value is either the time of the last contact with the quarry during the search or $-\infty$ if contact was never made. If temporarily we let $T = \infty$, then $R_1(t)$ can be viewed as the cumulative distribution function of the random variable X whose value gives the time between the beginning of the search and the pursuer's first contact with the quarry. Notice that $R_1(t)$ satisfies

$$1 - R_1(t+h) = [1 - R_1(t)][1 - R_1(h)] \quad t,h \geq 0$$

because the probability of not finding the quarry within t+h hours equals the probability of not finding the quarry within t hours times the probability of not finding the quarry within h hours. It is well known[2] that any cumulative distribution function that satisfies this equation must be of the form

$$R_1(t) = 1 - \exp(- t/L)$$

where L is the expected value of X; that is, L is the expected time until contact with the quarry is first made. Now remove the restriction that $T = \infty$. Then $R_1(t)$ takes the form

$$R_1(t) = \begin{cases} 1 - \exp(-t/L) & \text{for } t \in [0,T] \\ \text{undefined} & \text{otherwise} \end{cases} .$$

A similar argument applied to $R_2(t)$ would show

$$R_2(t) = \begin{cases} 1 - \exp(-t/M) & \text{for } t \in [0,T] \\ \text{undefined} & \text{otherwise} \end{cases} ,$$

where M is the expected length of time contact is continuously maintained with the quarry.

For nonnegative $t \leq T$, let $P(t)$ be the probability that the pursuer will have contact with the quarry at time t, given he had no contact at time 0. Also, define $S(t)$ to be the probability that the pursuer will make contact with the quarry before time t and thereafter not lose contact at least until time t, given he had no contact at time 0. We have immediately

$$R_1(t) \geq P(t) \geq S(t) \geq 0 \quad . \tag{1.1}$$

Notice that $S(t)$ satisfies

$$S(t) = \int_0^t [1 - R_2(t-x)] R_1'(x) dx \quad . \tag{1.2}$$

Since $R_1(0) = 0$, (1.1) implies

$$S(0) = P(0) = R_1(0) = 0 \quad . \tag{1.3}$$

Differentiating (1.2) and evaluating the derivative at 0 yields

$$S'(0) = R_1'(0) = 1/L \quad . \tag{1.4}$$

Equations (1.1), (1.3), and (1.4) imply

$$P(0) = R_1(0) = 0 \tag{1.5}$$

and

$$P'(0) = R_1'(0) = 1/L \quad . \tag{1.6}$$

Similarly, if we define $P_0(t)$ to be the probability that the pursuer will not have contact with the quarry at time t, given he had contact at time 0, then

$$P_0(0) = R_2(0) = 0 \tag{1.7}$$

and

$$P_0'(0) = R_2'(0) = 1/M \quad . \tag{1.8}$$

For any real h such that $0 < h \leq |T-t|$,

$$P(t+h) = P(t)[1 - P_0(h)] + [1 - P(t)]P(h) \quad .$$

Rearranging terms and dividing by h gives

$$\frac{P(t+h) - P(t)}{h} = \frac{P(h) - P(t)[P(h) + P_0(h)]}{h} \quad .$$

Also, if $0 < h < t$

$$\frac{P(t-h) - P(t)}{-h} = \frac{P(h) - P(t-h)[P(h) + P_0(h)]}{h} \quad .$$

The limits as $h \to 0$ of both these equations are equal, and thus the derivative of P(t) exists and equals

$$P'(t) = P'(0) - P(t)[P'(0) + P_0'(0)]$$

$$= 1/L - P(t)(1/L + 1/M) \quad .$$

The solution to this differential equation satisfying $P(0) = 0$ is

$$P(t) = \frac{M}{L + M}\left[1 - \exp\left(-\frac{L + M}{LM}t\right)\right] .$$

When necessary, to avoid ambiguity, we will write $P_{L,M}$ for P.

Now let $Q(t_0,t_1)$ be the probability that the pursuer will not have contact with the quarry in the time interval (t_0,t_1). Then

$$\begin{aligned} Q(t_0,t_1) &= [1 - P_{L,M}(t_0)][1 - P_{L,\infty}(t_1 - t_0)] \\ &= \frac{L\cdot\exp\left(-\frac{t_1 - t_0}{L}\right) + M\cdot\exp\left(-\frac{t_0}{M} - \frac{t_1}{L}\right)}{L + M} . \end{aligned} \quad (1.9)$$

Define the function F(t) from the real numbers to the closed unit interval as follows:

$$F(t) = \begin{cases} Q(0,T) & \text{for } t \le 0 \\ Q(t,T) & \text{for } t \in (0,T] \\ 1 & \text{for } t > T \end{cases} . \quad (1.10)$$

Then the probability that the last contact with the quarry before the end of the search takes place in the interval (t_1,t_2) is $F(t_2) - F(t_1)$, and the probability of never contacting the quarry is F(0). Therefore F(t) is the cumulative distribution function of the random variable Z, which is what we intended to derive.

Next we will derive the probability density function of the random vector W, which gives the vector difference between the location of the quarry at time T and the position determined by the pursuer at the time of the last contact. Assume the pursuer makes contact with the quarry at least once during the search, and let t denote the time elapsed between the last contact and the end of the search. Then the probability density

function f_t of W is just the convolution of h and g_t because W is the sum of the two independent random vectors U and V, and the probability density function of the sum of two independent random vectors is just the convolution of their probability density functions. Before deriving an expression for this convolution, first notice that if (r,θ) and (r',θ') are two vectors expressed in polar coordinates, then the length of their vector difference $(r,\theta) - (r',\theta')$ is

$$[r^2 + r'^2 - 2rr' \cos(\theta-\theta')]^{1/2} \quad .$$

The convolution $h{\star}g_t$ in polar coordinates is then

$$\begin{aligned} f_t(R,\theta) &= h{\star}g_t(R,\theta) \\ &= \int_0^\infty \int_0^{2\pi} h(r,\theta) g_t[(R,\theta) - (r,\gamma)] r\,d\gamma\,dr \\ &= \frac{1}{2\pi v} \int_0^{st} \int_0^{2\pi} \frac{1}{\pi(st)^2} \exp\left[- \frac{r^2 + R^2 - 2rR\cos(\gamma-\theta)}{2v}\right] r\,d\gamma\,dr \\ &= \frac{1}{2\pi^2 v(st)^2} \int_0^{st} \int_0^{2\pi} \exp\left[- \frac{r^2 + R^2 - 2rR\cos(\gamma)}{2v}\right] r\,d\gamma\,dr \ . \end{aligned} \tag{1.11}$$

This is as far as the model for the single-quarry operation will be developed. The two functions $f_t(R,\theta)$ and $F(t)$ are both needed for the multiple-quarry analysis, and, in fact, the purpose for studying the single-quarry case was specifically to derive expressions for them. In the next section, multiple-quarry searches will be decomposed into a number of simultaneous single-quarry searches, and the functions f_t and F will be applied to each search independently.

B. Multiple-Quarry Operation

Assume N quarries move on N unbounded euclidean planes with maximum speed s, one quarry to each plane. Assume also that no quarry has information about events outside his own plane. Suppose a pursuer will carry out N independent operations to locate each of the quarries. All of the operations will begin simultaneously and last for exactly T hours. At the end of the search, the pursuer will allocate his weapons against the N quarries and launch a simultaneous attack. His goal for the allocation will be to maximize the expected number of quarries destroyed.

We will assume that whenever the pursuer does not have contact with a quarry, the probability of detecting that quarry within any time interval will be a function of both the length of the interval and of the particular quarry involved. Denote the expected time to locate the i^{th} quarry by L_i. Similarly, whenever the pursuer has contact with a quarry, the probability of losing contact within any time interval will be a function of both the length of that interval and the quarry involved. Denote the expected time to lose contact with the i^{th} quarry by M_i. In the single-quarry analysis, we derived the expression (1.10) for F(t), the cumulative distribution function of the last time the pursuer had contact with the quarry during the search. F(t) is expressed in (1.10) in terms of the function $Q(t_1,t_2)$, which in turn is expressed in (1.9) in terms of L and M, the expected times to find and to lose the quarry. Since we now have N different quarries, each associated with different values of L and M, we need N different cumulative distribution functions, one for each quarry. We will denote the cumulative distribution function for the i^{th} quarry by $F_i(t)$. Then, if we substitute L_i and M_i for L and M in (1.9), $F_i(t)$ equals the right side of (1.10).

As in the single-quarry case, when the pursuer has contact with a quarry, the error distribution of his knowledge of that quarry's true position will be bivariate gaussian with covariance matrix vI. Also, if t denotes the number of hours between the last contact with a given quarry and the end of the search, then the probability distribution of that quarry's final position and his location at the time of last contact will be uniform within a disc of radius st. Therefore, the expression derived for $f_t(R,\theta)$ applies unchanged for each of the quarries.

Up to this point we have considered only the search phase of the pursuer's operation. After the search is concluded, however, the pursuer is still faced with the problem of allocating his weapons to maximize the expected number of quarries destroyed. Initially, we will assume that the pursuer's weapons are perfect in the sense that each weapon detonates exactly at its aimpoint and has probability 1 of killing a quarry within a radius k of its aimpoint and probability 0 elsewhere. We will also assume that the weapons detonate exactly at the conclusion of the search; that is, the times to allocate and to deliver the weapons are 0. Later, we will relax some of these restrictions. We will not attempt to develop a scheme for computing an exact maximal allocation, but instead we will develop a method of constructing a "good" allocation and an upper bound on how far the allocation is from the theoretical maximum.

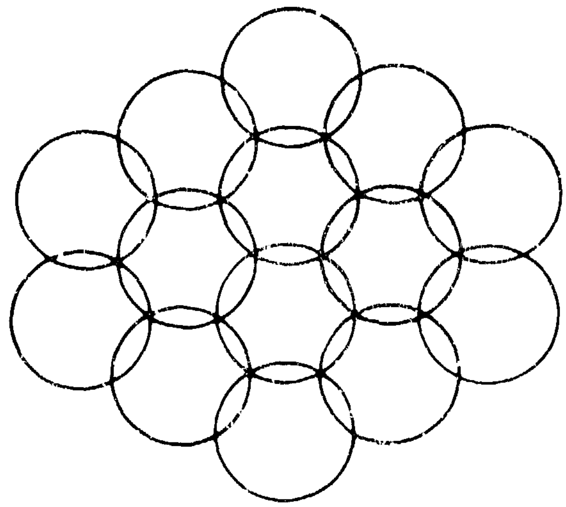
Assume the pursuer has W weapons and that the kill radius for each weapon is k. We may as well assume that he contacts at least one quarry during the search, otherwise the expected kill will be 0 irrespective of the allocation used. Suppose he contacts the i^{th} quarry. Denote the time between the last

contact with this quarry and the end of the search by t_i. Then $f_{t_i}(R,\theta)$ [defined by (1.11)] is the probability density function describing the vector difference between the i^{th} quarry's true position at the end of the search and the position determined by the pursuer at the time of his last contact with the quarry. f_{t_i} is independent of θ and is monotonically decreasing in R. If we were to allocate a_i weapons against this quarry, then a straightforward but somewhat technical Lebesgue measure theoretic argument (which we omit) would yield an upper bound on the probability of destroying the quarry of

$$\int_D\int f_{t_i}(r,\theta)r \;\; d\theta dr \quad , \tag{2.1}$$

where D is the disc of radius $k\sqrt{a_i}$ centered at the pursuer's estimate of the quarry's position at the time of the last detection. Notice that the area of D is just the total area that would be attacked by all the weapons if there were no overlap of their regions of coverage. Since it is impossible to cover a disc with smaller discs without overlap, (2.1) gives an unattainable upper bound.

In order to find a "good" attainable allocation, we need to look at the problem of covering a disc with smaller discs. Consider the covering that places the smaller discs in a regular hexagonal array within the larger disc so as to maximize the radius of the largest disc that can be covered by the pattern. We will call this pattern the hexagonal allocation. The pattern is illustrated by the diagram on the following page. It is easy to show that any covering disc that does not intersect the boundary of the larger disc loses $1 - 3\sqrt{3}/2\pi$ of its area to overlap. For a covering disc that does intersect the boundary, the fraction of its area lost to overlap is not so easily determined. Fortunately, as the



ratio of the radius of the larger disc to that of the smaller discs increases, the fraction of covering discs intersecting the boundary approaches 0. If this ratio is large enough, we can ignore these "edge effects" and approximate the number of discs needed to cover the larger disc by

$$(A/a)/(3\sqrt{3}/2\pi) \approx 1.21(A/a) \quad , \qquad (2.2)$$

where A is the area of the larger disc and a is the area of each covering disc.

Returning to the weapon allocation problem, if we accept the approximation (2.2) (which we will), then the hexagonal allocation using 1.21 a_i weapons will achieve at least as high a probability of kill as can be achieved with a_i weapons using the theoretically optimal allocation. Notice that because $f_{t_i}(R,\theta)$ is monotonically decreasing in R, if we remove 0.21 a_i of the 1.21 a_i discs farthest from the center of the allocation, we will have a probability of kill greater than 0.82 of that achieved by the theoretically best allocation of a_i weapons. But, because we are ignoring "edge

effects," and because the hexagonal allocation maximizes the radius of the largest disc that can be covered with a hexagonal array, we can achieve at least as high a probability of kill by using the hexagonal allocation for a_i weapons.

Now assume that at the end of the search, S quarries have been detected at least once; there is no loss of generality in labeling these quarries $1,2,\cdots,S$. For each $j = 1,2,\cdots,S$, define

$$f_j(r,\theta) = f_{t_j}(r,\theta) \quad .$$

Again, assume we will allocate a_i weapons against quarry i. Then we want to find $a_1,\cdots,a_S$, which maximizes

$$\sum_{i=1}^{S} \int_0^{k\sqrt{a_i}} \int_0^{2\pi} f_i(r,\theta) r \; d\theta dr \quad , \tag{2.3}$$

satisfying

$$a_i \geq 0 \quad i = 1,\cdots,S$$

and

$$\sum_{i=1}^{S} a_i = W \quad .$$

By Gibbs' Lemma[3] the solution $a_1,\cdots,a_S$ satisfies

$$f_i(k\sqrt{a_i},\theta) = d_0 \quad \text{if } f_i(0,\theta) \geq d_0$$

$$a_i = 0 \quad \text{if } f_i(0,\theta) < d_0$$

for some $d_0 > 0$.

The maximum value of (2.3) is an upper bound on the expected kill for any allocation. By allocating a_i weapons hexagonally to quarry i, we can achieve an expected kill greater than 0.82 of this upper bound.

We have brought the mathematical development of the model to the point where we can state the algorithm to which we

referred in the introduction, that is, an algorithm to compute the expected number of quarries destroyed as a function of the input variables. First, however, we will present a preliminary algorithm that will be used in the statement of the Main Algorithm. Assume the search phase of the operation is complete: let k, W, N, S, and $t_1, \cdots, t_N$ be defined as before. The following procedure will compute a weapon allocation that is at least 82 percent of optimal and will calculate the expected number of quarries destroyed (again, we are ignoring edge effects of the allocation).

ALGORITHM A

1. Set MX = 1, MN = 0, DN = 0.5
2. For $i = 1, 2, \cdots, S$, find r_i such that

   $$f_i(r_i, \theta) = DN \quad .$$

   If no such r_i exists, set $r_i = 0$.

   Set $B_i = 1.21(r_i/k)^2$.

   Set $B = B_1 + B_2 + \cdots + B_S$.
3. If B = W (within the desired accuracy), go to Step 7.
4. If B < W, go to Step 6.
5. Set MN = DN .
   Set DN = 0.5 (DN + MX) .
   Go to Step 2.
6. Set MX = DN .
   Set DN = 0.5 (DN + MN) .
   Go to Step 2.
7. Allocate B_i weapons to quarry i using the hexagonal allocation. The expected number of quarries destroyed is

$$K = \sum_{i=1}^{S} \int_0^{r_i} \int_0^{2\pi} f_i(r,\theta) \quad r d\theta dr \quad .$$

8. STOP.

The Main Algorithm, which we state below, is a Monte Carlo procedure that repeatedly applies Algorithm A to sets of times $t_1,\cdots,t_N$ that are chosen according to the distributions $F_1(t),\cdots,F_N(t)$. We state the algorithm in a general manner and leave out such details as stopping rules, methods of generating random variates, etc.

MAIN ALGORITHM

1. Set EK = 0, EKSQ = 0, KOUNT = 0 .
2. Obtain N random numbers $b_i, i = 1,\cdots,N$ from a population uniformly distributed in [0,1]. Set $t_i = T - F_i^{-1}(b_i)$.
3. Apply Algorithm A to obtain K.

   Set EK = EK + K, EKSQ = EKSQ + K^2, KOUNT = KOUNT + 1 .
4. Apply the stopping rule; if we are not finished, go to Step 2.
5. The expected number of quarries destroyed is

   $$E = EK/KOUNT \quad .$$

   The variance of the number of quarries destroyed is

   $$V = EKSQ/KOUNT - E^2 \quad .$$
6. STOP.

C. Notes

a. In the development of the Main Algorithm, we assumed the weapon delivery times were 0. The algorithm is easily modified to accommodate the case where these times are greater than 0.

If $d_1, \cdots, d_N$ are the delay times between the end of the search and the arrival of the weapons for quarries $1, \cdots, N$, respectively, then insert Step 2.5 into the algorithm, which adds d_i to t_i for each i.

b. If the weapon reliability p is less than unity, the allocation obtained by Algorithm A will not necessarily be "near optimal." The approach we used to obtain the algorithm, however, generalizes in a straightforward manner to include weapons with reliability less than 1. Algorithm A can be modified to handle such weapons by replacing Steps 2 and 7 with the following steps.

2'. For $i = 1,2,\cdots,S$ and $j = 1,2,\cdots,W$, find r_{ij} such that

$$p(1-p)^{j-1} f_i(r_{ij},\theta) = DN \quad .$$

If no such r_{ij} exists, set $r_{ij} = 0$. Set

$$B_i = 1.21 \sum_{j=1}^{W} \left(\frac{r_{ij}}{k}\right)^2$$

Set $B = B_1 + B_2 + \cdots + B_S$.

7'. Allocate B_i weapons to quarry i as follows. Find the aimpoints computed by the hexagonal allocation for $1.21(r_{i1}/\kappa)^2$ weapons. Then for each aimpoint whose distance from the origin is greater than or equal to $r_{i,j+1}$ and less than $f_{i,j}$ $(1 \le j < W)$, allocate j weapons. For each aimpoint between 0 and r_{iW}, allocate W weapons.

The expected number of quarries destroyed is

$$K = \sum_{j=1}^{W} p(1-p)^{j-1} \sum_{i=1}^{S} \int_{r_{i,j+1}}^{r_{ij}} \int_{0}^{2\pi} f_i(r,\theta) \quad rd\theta dr \quad ,$$

where $r_{i,W+1}$ is defined to be 0.

III. APPLICATION

In this part we will look at some of the characteristics of the model developed in Part II. We will do this by examining the results computed by a FORTRAN implementation of the Main Algorithm. The input parameters to this FORTRAN program were set to values that might approximate those corresponding to a nuclear barrage attack on a fleet of submarines. By varying the values of some of these parameters, we will be able to examine the sensitivity of the expected number of submarines destroyed to these values. The values of the parameters used in the example are purely hypothetical. No classified information concerning the characteristics of the detection system, weapons, or submarines has been used at all in developing the model or preparing this note.

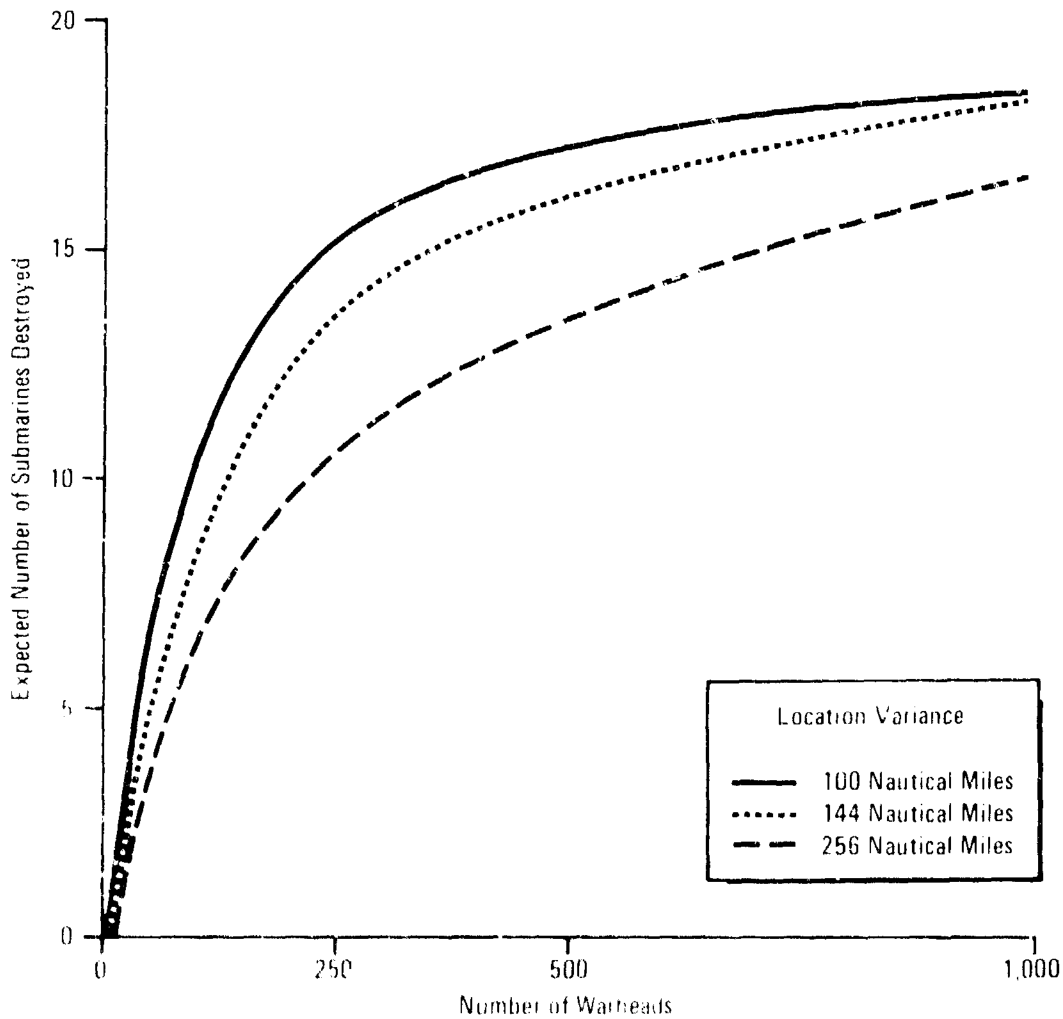
We are not going to do an in-depth analysis for this example. We will present graphs illustrating the sensitivity of the expected kill to only three parameters. Furthermore, we have only plotted enough data points to draw reasonable curves on these graphs. For any real application of the model the number of data points calculated would, of course, have to be chosen to insure the required accuracy.

To show the sensitivity of the expected kill to the three parameters assume a fleet of 20 submarines are at sea and that each vessel has a maximum speed of 6 knots. Assume also that an attacker with a force of nuclear missiles decides to attack the submarines. He will search for all the submarines independently and will apply an identical effort to find and trail each of them. At the end of 20 hours of searching he will launch an attack with his missiles. The missiles all reach their aimpoints 1 hour after the end of the search. Each missile has reliability 1 and will destroy any submarine within 6 nautical miles of its aimpoint.

Figure 1 plots the expected number of submarines destroyed against the number of warheads when the expected times to find and to lose each vessel are each 5 hours. The three curves on the graph represent three different variances of the attacker's locational system. For all three variances, the figure demonstrates that the more warheads available for the attack, the greater the expected number of vessels destroyed; it also demonstrates that the smaller the variance of the attacker's locational system, the higher the expected number of vessels destroyed. Finally, one can infer from the figure that as the number of warheads increases, the importance of the variance of the locational system to determining the expected kill decreases. That is, as the number of warheads increases, the absolute difference of the expected kill for any two variances decreases. If the three curves were continued to the right indefinitely, they would all asymptotically approach the same value, which is the expected number of submarines detected within the search period. Thus, at least theoretically, the attacker can always compensate for large locational uncertainties by increasing the number of warheads.

Figure 2 presents the same data as Figure 1 except that the ordinate is the variance of the location system. The three curves on the graph correspond to three different quantities of warheads. The asymptotic value for all three curves as the variance increases is 0. We can infer from the graph that as locational uncertainty of the attacker's locational system decreases, the improvement in the expected kill due to increasing the number of warheads diminishes. One might expect that when the variance reaches 0, all three curves would intersect. This is not necessarily true because, although there is no longer any uncertainty due to the locational system, there is still locational uncertainty due to

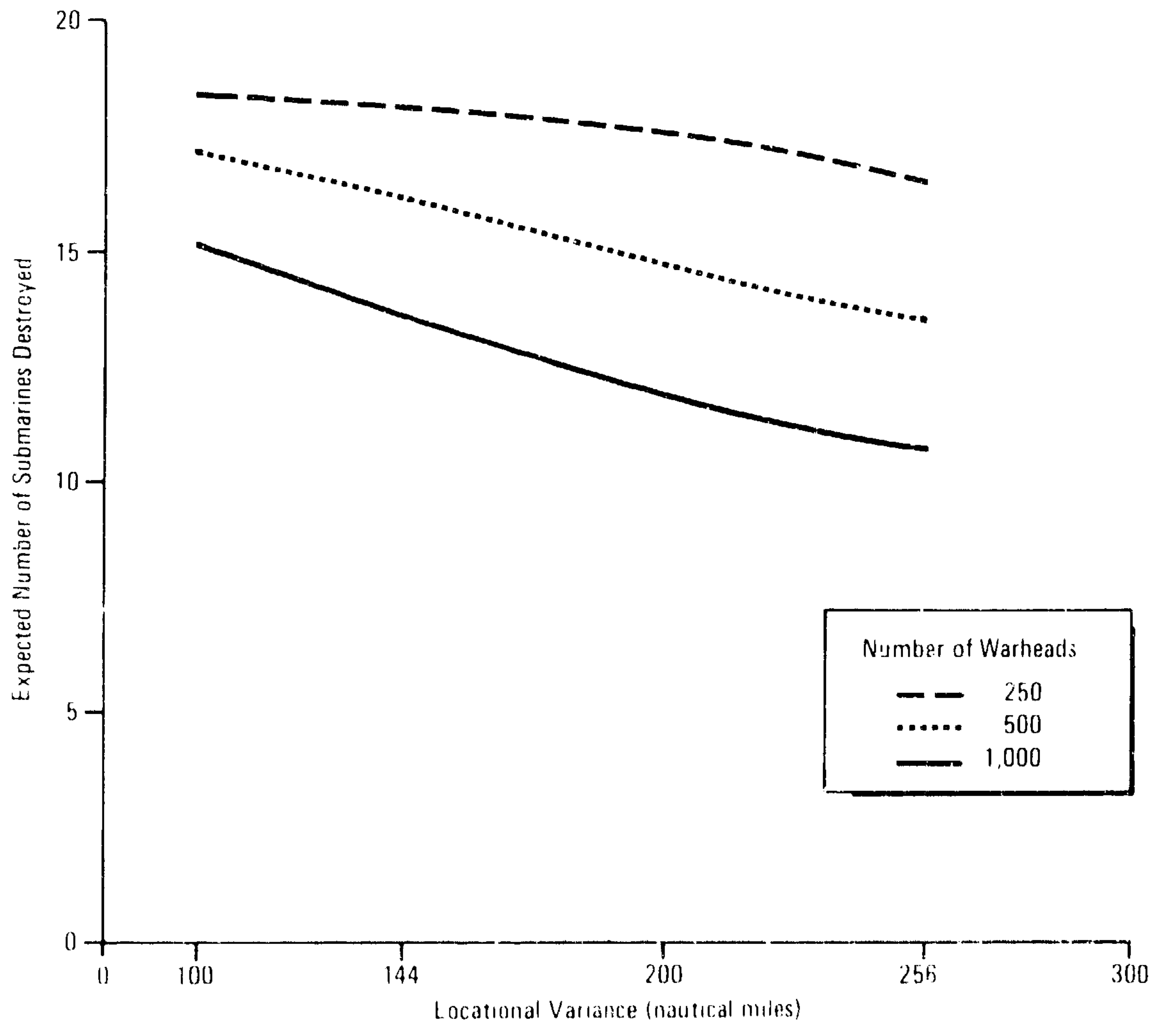
FIGURE 1
EXPECTED NUMBER OF SUBMARINES DESTROYED AS A FUNCTION OF THE NUMBER OF WEAPONS USED IN THE ATTACK



Notes:

- Number of Submarines—20
- Submarine Speed 6 Knots
- Weapon Kill Radius 6 Nautical Miles
- Expected Time to Find a Submarine 5 Hours
- Expected Time to Lose a Submarine 5 Hours
- Time From End of Search to Weapon Detonation 1 Hour
- Length of Search 20 Hours

FIGURE 2
EXPECTED NUMBER OF SUBMARINES DESTROYED AS A FUNCTION OF THE VARIANCE OF THE ATTACKER'S LOCATIONAL SYSTEM



Notes

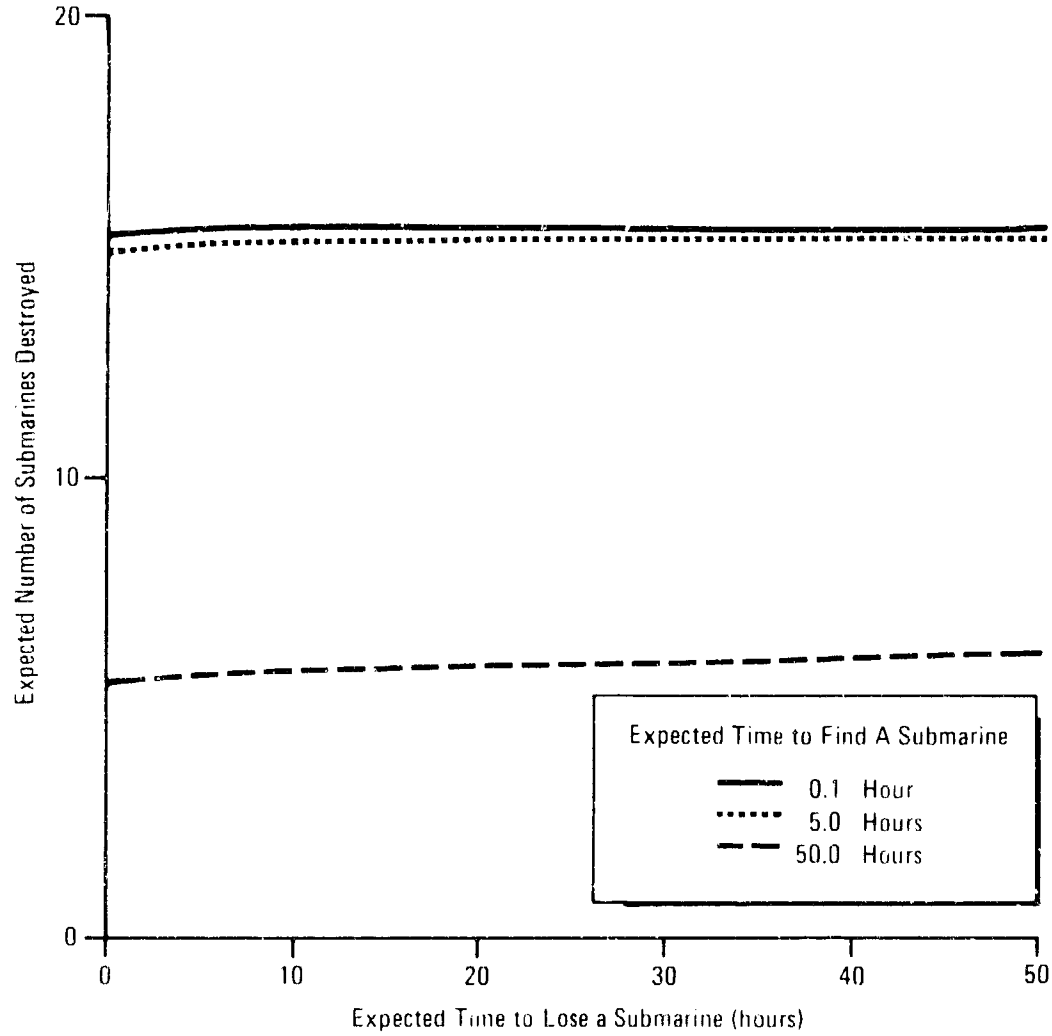
- Number of Submarines 20
- Submarine Speed 6 Knots
- Weapon Kill Radius 6 Nautical Miles
- Expected Time to Find a Submarine 5 Hours
- Expected Time to Lose a Submarine 5 Hours
- Time From End of Search to Weapon Detonation 1 Hour
- Length of Search 20 Hours

the submarine's random movement. It is true, however, that if the locational variance is 0, there is a maximum number of warheads, after which increasing that number will result in no further improvement to the expected kill. If the locational variance is greater than 0, however, the attacker can always improve the expected kill by increasing the number of warheads.

Figure 3 presents some very interesting and not altogether expected results. The figure plots the expected number of submarines destroyed against the expected time to lose contact with a vessel for three different values of the expected time to make contact. It is assumed the attacker's locational system has a variance of 100 nautical miles and that he has 250 weapons to allocate to the attack. The figure shows that the expected kill is almost independent of the attacker's ability to maintain contact with a submarine. If the attacker's only criterion for evaluating his search system is the expected number of vessels it will allow him to destroy, then it is a waste of effort for him to improve this aspect of his search system, as such an improvement would make virtually no difference on the expected outcome of the attack. Notice, however, that the expected outcome is considerably more sensitive to the attacker's ability to contact a submarine. That is, although the expected time to lose a vessel is unimportant in determining the expected kill, the expected time to find a vessel is an important parameter.

In this example, the variance of the attacker's locational system was chosen to be large with respect to the submarine speed. The distance a submarine could be expected to move between the time the attacker loses and reacquires contact is small relative to this variance and does not contribute much to decreasing the effectiveness of the attack. Therefore, Figure 3 does not demonstrate an inherent characteristic of

FIGURE 3
EXPECTED NUMBER OF SUBMARINES DESTROYED
AS A FUNCTION OF THE EXPECTED LENGTH OF TIME
THE ATTACKER MAINTAINS UNINTERRUPTED CONTACT



Notes:

- Number of Submarines–20
- Submarine Speed–6 Knots
- Weapon Kill Radius–6 Nautical Miles
- Time From End of Search to Weapon Detonation–1 Hour
- Number of Weapons–250
- Locational Variance–100 Nautical Miles
- Length of Search–20 Hours

the model, but only a characteristic of the model applied to this particular example.

REFERENCES

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This Division Note presents a mathematical model of an operation in which a pursuer attempts to locate and destroy a group of moving quarries. The operation consists of a search lasting for a predetermined length of time, followed by a near-optimal attack. Characteristics of the attacker's locational and weapon systems and of the quarries are used as inputs to the model, which calculates the expected outcome and the variance of the outcome of the operation.

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